Unit 1  Mechanics of sediment transportation

Shinji Egashira  
Dr. Eng., Executive Technology Advisor, NEWJEC Inc.  
Guest Professor, National Graduate Research Institute for Policy Studies  
E-mail: egashirasn@newjec.co.jp; segashira@fm2.seikyou.ne.jp  
Address: Honjo-higashi 2-3-20, Osaka 531-0074, Japan

64 Years Old (Feb. 15, 1947)

1973 Ms. Graduate School of Kyoto University
1973 Research Associate, Disaster Prevention Research Institute, Kyoto University
1980 Dr. Eng. (Kyoto University)
1982 Associate Professor, Disaster Prevention Research Institute, Kyoto University
1991 Visiting Professor, University of Minnesota
1994-2007 Professor of Ritsumeikan University
2007-Present Executive Technology Advisor, NEWJEC Inc.
2008-Present Guest Professor, National Graduate Research Institute for Policy Studies

1973-1980: Density currents, Dynamics of Reservoir Water, Sediment-induced disasters  
1980- Sediment yield and transportation, Sediment induced disasters  
1985- Sediment transportation, River morphology  
1988- Debris flow

Unit-1 Mechanics of sediment transportation

1. Introduction

It is my memory that subjects such as mechanics, dynamics, physics etc. provided in the beginning stage, 1-2 years of university could be learned naturally.

However, subjects associated with sediment transportation may be difficult even for graduate students. I remembered it was uneasy for me to understand sediment hydraulics when I was student. It may be even now for me. One of the reason why kinematics of sediment particles such as rolling, sliding, jumping and suspension are usually discussed based on their own experiences and are treated without dynamic principles, supposing:

- velocity of bed sediment is proportional to ---
- step length is proportional to ---
- bed load is composed of sediment particles in bed surface layer---

Bed surface layer? Definition of bed surface?---Many and many questions appear.
Where is the bed surface?

(a) Sediment concentration $c=0$ by volume
(b) Sediment concentration $c=c^*$ and sediment particles are stationary
(c) Sediment concentration $c=c$ and sediment particles are carried by turbulent suspension.

Fig. Pore-water pressure

\[ p = \rho gh \]
\[ p = \rho gh \]
\[ p = \rho_m gh \]
\[ \rho_m = \sigma c + (1-c) \rho \]
2. bed load formulas

2.1 Brief review

Important parameters:

The non-dimensional bed shear stress

\[ \tau^* = \frac{u^2}{(\sigma/\rho - 1)gd} = \frac{hi_e}{(\sigma/\rho - 1)d} \]

\[ u_s = \sqrt{\tau^*/\rho} \]: Shear velocity

\[ d \]: sediment particle size

\[ \rho \]: mass density of water

\[ \sigma \]: mass density of sediment particle

Sediment transport forms

How is the pore water pressure at each bed surface?
Meyer-Peter and Muller proposed based on dimensional consideration,

\[ q_{bs} = 8 (\tau_* - 0.047)^{3/2} \]  

(1)

\[ q_{bs} : \text{non-dimensional bed load transport rate defined as} \]

\[ q_{bs} = q_b / \sqrt{(\sigma / \rho - 1)gd^3} \]

Still now, Meyer-Peter and Muller’s equation is widely used.

Since the publication of this equation, many formulas have been proposed. Their methods can be classified as

Bed load rate = [pick up rate]^p[step length] \hspace{1cm} (2a)

Bed load rate = [sediment volume]^p[velocity] \hspace{1cm} (2b)

Einstein(1950) proposed, based on stochastic method (Eq. 2a)

\[ \frac{A_b q_{bs}}{1 - A_b q_{bs}} = 1 - \frac{1}{\sqrt{\pi}} \int_{-B* \tau_*^{-1} / \eta_*}^{B* \tau_*^{-1} / \eta_*} e^{-t^2} dt \]  

(3)

\[ \eta_* = 0.5 \quad A_* = 27.0 \quad B_* = 0.156 \]

In 60’s and 70’s, studies based on stochastic methods were conducted actively (i.e. Yang and Sayer 1971, Nakagawa and Tsujimoto 1980).

Simultaneously, studies based on Eq. (2b) were conducted (Bagnold 1954, 1957, 1966). The bed load transport rate corresponding to Eq. (2b) can be described:

\[ q_b = V_s u_s = \int_0^{h_b} cudz = \bar{c}_s \bar{h}_s u_s \]  

(4)

Bagnold’s type bed load formulas are described as

\[ q_b \sim \tau_b u \sim \tau_b^{3/2} \]
Bagnold introduced grain shear stress described by

\[ \tau_G = \sigma_n \tan \alpha \quad (\sigma_n = (\sigma - \rho)gV_s \cos \theta) \quad (5) \]

\( \sigma_n \) : Grain normal stress to bed surface

Referring to bed shear stress supposed by them, it is considered that the structure of their bed load equations is similar to each other.

For instance, Ashida et al. described grain shear stress (Eq. 5) as

\[ \tau_G = (\sigma - \rho)gV_s \cos \theta \mu_f \quad (6) \]

\( \tau_c \) : critical bed shear stress, \( \mu_f \) friction coefficient, \( V_s \) : absolute sediment volume of bed load layer in unit area.

In addition, they evaluated the velocity of bed load layer using mass point system for a single grain;

\[ u_s \sim u_a \quad (7) \]

The bed shear stress is

\[ \tau_b = \tau_G + \tau_c \quad (8) \]
Ashida et al. determined a functional form of the bed load equation using Eqs. (6), (7) and (8) and proposed the following form through calibration process.

\[ q_{bs} = 17\tau_{se}^{3/2} \left( \frac{1 - \frac{\tau_{se}}{\tau_s}}{1 - \frac{u_{se}}{u_s}} \right) \]  

\[ q_{bs} = \frac{q_b}{\sqrt{(\sigma/\rho - 1)gd^3}} \]

\( \tau_{se} \): effective bed shear stress defined by \( \tau_{se} = u_{se}^2/[(\sigma/\rho - 1)gd] \), in which effective shear velocity should be computed by

\[ \frac{v}{u_{se}} = 6.0 + 2.5\ln\frac{h}{d(1 + 2\tau_s)} \]

\[ q_b = V_s u_s = \int_0^{h_s} cudz = \bar{c}_s h_s u_s \]  

In these treatments the control volume to discuss stress structure was not defined, and thus ……
2.2 Bed load formula derived by constitutive relations

Egashira, Miyamoto and Itoh (1997) extended their debris flow model to bed load layer in order to solve the velocity profile.

\[
\tau = p_s \tan \phi_s + k_d (1 - e^2) c^{1/3} d^2 \left( \frac{\partial u}{\partial z} \right)^2 + k_f \left( \frac{1 - c}{c^{2/3}} \right)^{5/3} d^2 \left( \frac{\partial u}{\partial z} \right)^2
\]  \hspace{1cm} (11)

\[
p = p_s + p_d + p_w
\]

\[
\frac{p_s}{p_s + p_d} = \left( \frac{c}{c_*} \right)^{1/n}, \quad (n=5)
\]

\[
p_d = k_d \sigma^2 c^{1/3} d^2 \left( \frac{\partial u}{\partial z} \right)^2
\]

\[
p_w = \rho g (h_t - z) \cos \theta
\]  \hspace{1cm} (12)

Thickness of bed load layer

\[
\tau_y(z) + \tau_s(z) + \tau_f(z) = \rho g \sin \theta \int_0^z \left( \frac{c}{c_*} \right)^{1/3} d z
\]  \hspace{1cm} (13)

\[
p_s(z) + p_d(z) = \rho g \cos \theta \int_0^z \left( \frac{c}{c_*} \right)^{1/3} d z
\]  \hspace{1cm} (14)

\[
\tau_y(0) + \tau_s(0) + \tau_f(0) = \rho g \sin \theta \int_0^u \left( \frac{c}{c_*} \right)^{1/3} d z + \int_0^u \left( \frac{c}{c_*} \right)^{1/3} d z
\]  \hspace{1cm} (15)

\[
p_s(0) + p_d(0) = \rho g \cos \theta \int_0^u \left( \frac{c}{c_*} \right)^{1/3} d z + \int_0^u \left( \frac{c}{c_*} \right)^{1/3} d z
\]  \hspace{1cm} (16)
Eqs. (15) and (16) yield
\[
\frac{h_s}{h_t} = \frac{1}{\left(\frac{\sigma}{\rho} - 1\right)\bar{c}_s} \tan \phi_s - \tan \theta
\]
\[
\frac{h_s}{d} = \frac{1}{\bar{c}_s \cos \theta} \tan \phi_s - \tan \theta
\]

\[
\tau_s = \int_0^{h_s} c_d z / h_s \approx c_0 / 2
\]
\[
\tau_s(z) + \tau_d(z) + \tau_f(z) = \rho g \sin \theta \left[ \int_z^{h_s} \left( \frac{\sigma}{\rho} - 1 \right) c + 1 \right] dz + \int_0^{h_s} dz \\
p_s(z) + p_d(z) = \rho g \cos \theta \left[ \int_z^{h_s} \left( \frac{\sigma}{\rho} - 1 \right) c dz + \int_0^{h_s} 0 dz \right]
\]

Substitution of constitutive relations into these two equations yield velocity distribution and the average velocity of the bed load layer:

\[
\frac{\tau_s}{u_*} = \frac{4}{15} \frac{K_1 K_2}{\sqrt{f_d + f_f}} \tau_*, \quad (18) \quad K_1 = \frac{1}{\cos \theta \tan \phi_s - \tan \theta} \\
u_* = \sqrt{gh_s \sin \theta} \quad K_2 = \frac{1}{c_s} \left[ 1 - \frac{h_s}{h_{ls}} \right]^{1/2} \\
\tau_s = u_*^2 / (\sigma - 1) g d \\
f_d = k_d \left( 1 - e^2 \right) |\sigma| \rho c_s \tau_0^{1/3} \\
f_f = k_f \left( 1 - c_s \right)^{5/3} c_s^{-2/3} \\
c_s = \int_0^{h_s} c dz / h_s \approx c_* / 2
\]

Bed load formula can be derived:

\[
q_b = \int_0^{h_s} c u dz = \bar{c}_s h_s u_* \\
\frac{\tau_s}{u_*} = \frac{4}{15} \frac{K_1 K_2}{\sqrt{f_d + f_f}} \tau_* \\
\frac{h_s}{d} = \frac{1}{\bar{c}_s \cos \theta \tan \phi_s - \tan \theta} \tau_* \\
q_{b^*} = \frac{4}{15} \frac{K_1^2 K_2}{\sqrt{f_d + f_f}} \tau_*^{5/2} \quad (19)
\]
3. Transportation of suspended sediment

Usually, mass conservation equation of suspended sediment is described as

\[
\frac{\partial c}{\partial t} + \frac{\partial c u}{\partial x} + \frac{\partial c v}{\partial y} + \frac{\partial (c w - c w_0)}{\partial z} = \frac{\partial}{\partial x} \left( \varepsilon \frac{\partial c}{\partial x} \right) + \frac{\partial}{\partial y} \left( \varepsilon \frac{\partial c}{\partial y} \right) + \frac{\partial}{\partial z} \left( \varepsilon \frac{\partial c}{\partial z} \right) \tag{20}
\]

Eq.(20) describes only settling processes of sediment particles.
Can this equation explain the lifting process of sediment?

Supposing steady- uniform flow and equilibrium sediment transportation, following results are obtained.

\[
\frac{\partial c}{\partial t} + \frac{\partial c u}{\partial x} + \frac{\partial c v}{\partial y} + \frac{\partial (c w - c w_0)}{\partial z} = \frac{\partial}{\partial x} \left( \frac{\partial c}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial c}{\partial y} \right) + \frac{\partial}{\partial z} \left( \frac{\partial c}{\partial z} \right)
\]

\[ (20) \]

\[
\epsilon_z \frac{dc}{dz} + w_0 c = 0 \Rightarrow \frac{c(z)}{c_{ae}} = \left( \frac{h - z}{z} \right)^A \quad A = w_0 \ell f_k u^* \]

\[ c_{ae} \] : equilibrium sediment concentration at a reference level.

The depth integrated form of Eq. (20) is:

\[
\frac{\partial \bar{c}_h}{\partial t} + \frac{\partial \bar{c} \bar{u}_h}{\partial x} + \frac{\partial \bar{c} \bar{v}_h}{\partial y} = \frac{\partial}{\partial x} \left( \bar{h} \frac{\partial \bar{c}}{\partial x} \right) + \frac{\partial}{\partial y} \left( \bar{h} \frac{\partial \bar{c}}{\partial y} \right) + E - D
\]

\[ (21) \]

\[
r_i = \int \sqrt{\bar{c}} dz / \sqrt{\bar{c} \bar{h}} \quad E : \text{Erosion rate} \quad D : \text{Deposition rate} \quad \gamma = 1 \quad \text{for wash load due to homogeneous sediment concentration}
\]

Erosion and deposition rates

Deposition rate is usually evaluated in terms of particle fall velocity, \( w_0 \) and sediment concentration, \( c_b \) in the vicinity of bed.

\[
D = w_0 c_b
\]

\[ (22) \]

The erosion rate is evaluated usually using \( c_{ae} \)

\[
E = w_0 c_{ae}
\]

\[ (23) \]
Evaluation of $c_{ae}$

Equation of motion of mass point system in vertical direction is used (i.e. Itakura & Kishi 1980):

\[
\frac{d}{dt} (mv) = F - G \\
\int_0^{mv_0} d(mv) = \int_0^t (F - G) dt
\]

Upwards flux $= W_o c_{ae} \quad \Rightarrow \quad c_{ae}$

Where is the reference level?
Concluding remarks (suggestions or my issues) for future studies

- Please be mindful for differences of kinematic and dynamic conditions for the bed surfaces between rigid and erodible beds.
  - definition of the bed surface (shear stress)
  - velocity and velocity gradient
  - sediment concentration
  - reference level of …..

- Erosion rate or equilibrium sediment concentration at a reference level is recommended to study in relation to the bed load layer.

- It is expected classic topics such as bed load formulas and suspended sediment transportation are studied again.